SUPPLEMENTARY MATERIALS: SAFE RULES FOR THE IDENTIFICATION OF ZEROS IN THE SOLUTIONS OF THE SLOPE PROBLEM*

CLÉMENT ELVIRA† AND CÉDRIC HERZET‡

SM1. Connection with existing safe screening test for SLOPE.

In this section, we show that the result given in Theorem 4.1 is *equivalent* (in a sense that we will make clear in Proposition SM1.1 below) to the screening test derived in [SM1, Proposition 1]. We organize our exposition as follows. Our equivalence result is formulated and proved in Subsection SM1.1. Sections SM1.2 and SM1.3 are dedicated to the proof of technical lemmas.

SM1.1. Equivalence of the results.

For self-containedness, we first rephrase the screening test proposed by Larsson and coauthors and reproduce a part of the result given in Theorem 4.1.

Denote $\mathbf{g}^\star \triangleq \lambda^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star$ where $\mathbf{u}^\star$ is the solution of the dual problem of SLOPE (see (4.4)), and let $\sigma\colon [\![1,n]\!] \to [\![1,n]\!]$ be a permutation of $[\![1,n]\!]$ such that $\mathbf{g}^\star_{[\ell]} = \mathbf{g}^\star_{(\sigma(\ell))}$ $\forall \ell \in [\![1,n]\!]$. The screening test proposed by Larsson *et al.* in [SM1, Proposition 1] can then be rephrased as follows: let $q_0 \in [\![1,n+1]\!]$,

$$\text{(SM1.1)}\qquad \begin{cases} \displaystyle\sum_{k=1}^{q_0-1} |\mathbf{g}^\star|_{[k]} = \sum_{k=1}^{q_0-1} \gamma_k \\ \displaystyle\forall q \in [\![q_0,n]\!]:\ \sum_{k=q_0}^{q} |\mathbf{g}^\star|_{[k]} < \sum_{k=q_0}^{q} \gamma_k \end{cases} \implies \quad \forall \ell \in [\![q_0,n]\!]:\ \mathbf{x}^\star_{(\sigma(\ell))} = 0.$$

On the other hand, our result in Theorem 4.1 reads as: let $\ell \in [\![1,n]\!]$,

$$\text{(SM1.2)}\qquad \forall q \in [\![1,n]\!]:\ |\mathbf{g}^\star_{(\ell)}| + \sum_{k=1}^{q-1} |\mathbf{g}^\star_{\backslash \ell}|_{[k]} < \sum_{k=1}^{q} \gamma_k \implies \mathbf{x}^\star_{(\ell)} = 0.$$

The equivalence between these two tests is encapsulated in the following proposition:

Proposition SM1.1 (Equivalence of tests (SM1.1) and (SM1.2)). *The left-hand side of* (SM1.1) *holds for some* $q_0 \in [\![1,n]\!]$ *if and only if the inequalities in* (SM1.2) *are verified* $\forall \ell:\ \sigma(\ell) \geq q_0$.

To simplify our exposition, we suppose (without loss of generality[1]) that

$$\text{(SM1.3)}\qquad \sigma(\ell) = \ell \quad \forall \ell \in [\![1,n]\!]$$

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The research presented in this paper is reproducible. Code and data are available at https://gitlab-research.centralesupelec.fr/2020elvirac/slope-screening

† IETR UMR CNRS 6164, CentraleSupelec Rennes Campus, 35576 Cesson Sévigné, France (clement.elvira@centralesupelec.fr, https://c-elvira.github.io/).
‡Inria centre Rennes - Bretagne Atlantique, Rennes, France (cedric.herzet@inria.fr, http://people.rennes.inria.fr/Cedric.Herzet/).

[1]This setup can always be attained by a proper reordering of the columns of $\mathbf{A}$.

that is

$$|\mathbf{g}^\star_{(\ell)}| = |\mathbf{g}^\star|_{[\ell]} \quad \forall \ell \in [\![1, n]\!]. \tag{SM1.4}$$

We divide our proof into two parts corresponding to the statements of the following lemmas:

LEMMA SM1.2. *The left-hand side of (SM1.1) holds for no $q_0 \in [\![1, n]\!]$ if and only if the left-hand side of (SM1.2) holds for no $\ell \in [\![1, n]\!]$.*

LEMMA SM1.3. *If the left-hand side of (SM1.1) holds for some $q_0 \in [\![1, n]\!]$ then test (SM1.2) passes $\forall \ell \in [\![q_0, n]\!]$ and fails $\forall \ell \in [\![1, q_0 - 1]\!]$. Conversely, if test (SM1.2) passes for some indices in $[\![1, n]\!]$ then the set of indices passing (SM1.2) necessarily takes the form $[\![q_0, n]\!]$ for some $q_0 \in [\![1, n]\!]$. Moreover, test (SM1.1) is passed for q_0.*

Lemma SM1.2 ensures that (SM1.1) and (SM1.2) identify no zeros of $\mathbf{x}^\star$ under exactly the same condition. Lemma SM1.3 shows that if some zeros are identified by one of the tests, the other test identifies exactly the same set of elements. The proofs of these two results are provided below in Sections SM1.2 and SM1.3, respectively.

SM1.2. Proof of Lemma SM1.2. The statement of Lemma SM1.2 can be rephrased mathematically as:

$$\sum_{k=1}^{n} |\mathbf{g}^\star|_{[k]} = \sum_{k=1}^{n} \gamma_k \iff \forall \ell \in [\![1, n]\!], \exists q_\ell \in [\![1, n]\!] : \ |\mathbf{g}^\star_{(\ell)}| + \sum_{k=1}^{q_\ell - 1} |\mathbf{g}^\star_{\setminus \ell}|_{[k]} = \sum_{k=1}^{q_\ell} \gamma_k. \tag{SM1.5}$$

We next show that the direct and converse of implications hold.

If the left-hand side of (SM1.5) holds, it is easy to see that $\forall \ell \in [\![1, n]\!]$, an equality occurs in the right-hand side for $q_\ell = n$.

Conversely, assume that the right-hand side of (SM1.5) holds. Consider $\ell = n$ and let $q_n \in [\![1, n]\!]$ be such that

$$|\mathbf{g}^\star_{(n)}| + \sum_{k=1}^{q_n - 1} |\mathbf{g}^\star_{\setminus n}|_{[k]} = \sum_{k=1}^{q_n} \gamma_k. \tag{SM1.6}$$

If $q_n = n$ then the result immediately follows. If $q_n < n$, the result can be proved by showing that (SM1.6) implies

$$\forall k \in [\![q_n, n]\!] : \ |\mathbf{g}^\star|_{[k]} = \gamma_k = \gamma, \tag{SM1.7}$$

for some constant γ. Indeed, if (SM1.7) holds we easily obtain the desired implication since

$$\sum_{k=1}^{n} |\mathbf{g}^\star|_{[k]} = \sum_{k=1}^{q_n-1} |\mathbf{g}^\star|_{[k]} + |\mathbf{g}^\star|_{[n]} + \sum_{k=q_n}^{n-1} |\mathbf{g}^\star|_{[k]} \tag{SM1.8}$$

$$= \sum_{k=1}^{q_n-1} |\mathbf{g}^\star_{\backslash n}|_{[k]} + |\mathbf{g}^\star_{(n)}| + \sum_{k=q_n}^{n-1} |\mathbf{g}^\star|_{[k]} \tag{SM1.9}$$

$$= \sum_{k=1}^{q_n} \gamma_k + \sum_{k=q_n}^{n-1} |\mathbf{g}^\star|_{[k]} \tag{SM1.10}$$

$$= \sum_{k=1}^{n} \gamma_k \tag{SM1.11}$$

where we used (SM1.4) for index $\ell = n$ in (SM1.9), (SM1.6) in (SM1.10) and (SM1.7) in (SM1.11). Hereafter, we thus concentrate on the proof of (SM1.7).

Assume that $q_n < n$. Let first note that, if (SM1.6) is verified, then the following series of inequalities hold $\forall q \in [\![q_n, n]\!]$:

$$\begin{aligned}\sum_{k=1}^{q_n} \gamma_k &= |\mathbf{g}^\star_{(n)}| + \sum_{k=1}^{q_n-1} |\mathbf{g}^\star_{\backslash n}|_{[k]} \\ &\leq |\mathbf{g}^\star|_{[q]} + \sum_{k=1}^{q_n-1} |\mathbf{g}^\star|_{[k]} \\ &\leq |\mathbf{g}^\star|_{[q_n]} + \sum_{k=1}^{q_n-1} |\mathbf{g}^\star|_{[k]} \\ &\leq \sum_{k=1}^{q_n} \gamma_k. \end{aligned} \tag{SM1.12}$$

The equality follows from (SM1.6) and the first two inequalities from (SM1.4). The last inequality can be obtained by noting that $\mathbf{g}^\star \in \partial r_{\text{SLOPE}}(\mathbf{x}^\star)$ by virtue of standard optimality conditions, see Lemma A.1; using the expression of the subdifferential of $\partial r_{\text{SLOPE}}(\mathbf{x}^\star)$ in Lemma A.3, this can also be rewritten as

$$\forall q \in [\![1, n]\!] : \sum_{k=1}^{q} |\mathbf{g}^\star|_{[k]} \leq \sum_{k=1}^{q} \gamma_k. \tag{SM1.13}$$

The last inequality in (SM1.12) is thus a consequence of (SM1.13) with $q = q_n$.

We note that since the first and last terms in (SM1.12) are the same, equality must hold throughout the expression. In particular, we must have

$$\forall q \in [\![q_n, n]\!] : |\mathbf{g}^\star|_{[q]} = |\mathbf{g}^\star|_{[n]}. \tag{SM1.14}$$

This shows the first part of (SM1.7).

The equality of the weights γ_k's in (SM1.7) can be shown as follows. On the one hand, repeating the same arguments as in (SM1.8)-(SM1.10) and since $q_n < n$ by hypothesis, we easily obtain that

$$\sum_{k=1}^{q_n+1} |\mathbf{g}^\star|_{[k]} = \sum_{k=1}^{q_n} \gamma_k + |\mathbf{g}^\star|_{[q_n+1]}. \tag{SM1.15}$$

Since $\sum_{k=1}^{q_n+1} |\mathbf{g}^\star|_{[k]} \leq \sum_{k=1}^{q_n+1} \gamma_k$ from (SM1.13), this leads to

$$|\mathbf{g}^\star|_{[q_n+1]} \leq \gamma_{q_n+1}. \tag{SM1.16}$$

On the other hand, we also have from (SM1.13):

$$\sum_{k=1}^{q_n-1} |\mathbf{g}^\star|_{[k]} \leq \sum_{k=1}^{q_n-1} \gamma_k. \tag{SM1.17}$$

Since equality holds in (SM1.12), this leads to

$$\sum_{k=1}^{q_n} \gamma_k = |\mathbf{g}^\star|_{[q_n]} + \sum_{k=1}^{q_n-1} |\mathbf{g}^\star|_{[k]} \leq |\mathbf{g}^\star|_{[q_n]} + \sum_{k=1}^{q_n-1} \gamma_k,$$

or more simply

$$\gamma_{q_n} \leq |\mathbf{g}^\star|_{[q_n]}. \tag{SM1.18}$$

Finally, combining (SM1.14), (SM1.16) and (SM1.18), we obtain

$$\gamma_{q_n} \leq |\mathbf{g}^\star|_{[q_n]} = \mathbf{g}^\star|_{[q_n+1]} \leq \gamma_{q_n+1}. \tag{SM1.19}$$

Since $\gamma_{q_n} \geq \gamma_{q_n+1}$ by definition, equality must hold throughout (SM1.19) and thus $\gamma_{q_n} = \gamma_{q_n+1}$. The same result can be obtained for any $q \in [\![q_n, n-1]\!]$ by repeating the arguments recursively.

SM1.3. Proof of Lemma SM1.3. (⇒) Assume that the left-hand side of (SM1.1) holds for some $q_0 \in [\![1, n]\!]$ that is $\exists q_0 \in [\![1, n]\!]$ such that

$$\sum_{k=1}^{q_0-1} |\mathbf{g}^\star|_{[k]} = \sum_{k=1}^{q_0-1} \gamma_k \tag{SM1.20}$$

$$\forall q \in [\![q_0, n]\!] : \sum_{k=q_0}^{q} |\mathbf{g}^\star|_{[k]} < \sum_{k=q_0}^{q} \gamma_k. \tag{SM1.21}$$

We now show that entries ℓ passes test (SM1.2) if and only if $\ell \in [\![q_0, n]\!]$.[2]

Let us first consider $\ell < q_0$ and show that ℓ does not pass test (SM1.2). More specifically, we emphasize that the inequality corresponding to $q = q_0 - 1$ is violated

[2]We remind the reader that we assume that (SM1.4) holds to simplify our derivations.

in the left-hand side of (SM1.2):

$$|\mathbf{g}^\star_{(\ell)}| + \sum_{k=1}^{q_0-2} |\mathbf{g}^\star_{\backslash \ell}|_{[k]} = |\mathbf{g}^\star|_{[\ell]} + \sum_{k=1}^{q_0-2} |\mathbf{g}^\star_{\backslash \ell}|_{[k]} \tag{SM1.22}$$

$$= \sum_{k=1}^{q_0-1} |\mathbf{g}^\star|_{[k]} \tag{SM1.23}$$

$$= \sum_{k=1}^{q_0-1} \gamma_k \tag{SM1.24}$$

where the first equality follows from (SM1.4), the second from $\ell < q_0$ and the last from (SM1.20).

Let us next show that test (SM1.2) passes when $\ell \geq q_0$, that is the inequalities in the left-hand side of (SM1.2) are verified $\forall q \in [\![1, n]\!]$. On the one hand, if $q \geq q_0$ we have

$$|\mathbf{g}^\star_{(\ell)}| + \sum_{k=1}^{q-1} |\mathbf{g}^\star_{\backslash \ell}|_{[k]} \leq \sum_{k=1}^{q} |\mathbf{g}^\star|_{[k]} \tag{SM1.25}$$

$$= \sum_{k=1}^{q_0-1} |\mathbf{g}^\star|_{[k]} + \sum_{k=q_0}^{q} |\mathbf{g}^\star|_{[k]} \tag{SM1.26}$$

$$< \sum_{k=1}^{q_0-1} \gamma_k + \sum_{k=q_0}^{q} \gamma_k = \sum_{k=1}^{q} \gamma_k \tag{SM1.27}$$

where (SM1.27) follows from (SM1.20)-(SM1.21). On the other hand, if $q < q_0$, we can write:

$$|\mathbf{g}^\star_{(\ell)}| + \sum_{k=1}^{q-1} |\mathbf{g}^\star_{\backslash \ell}|_{[k]} \leq |\mathbf{g}^\star|_{[\ell]} + \sum_{k=1}^{q-1} |\mathbf{g}^\star|_{[k]} \tag{SM1.28}$$

$$\leq |\mathbf{g}^\star|_{[q_0]} + \sum_{k=1}^{q-1} |\mathbf{g}^\star|_{[k]} \tag{SM1.29}$$

$$< \gamma_{q_0} + \sum_{k=1}^{q-1} \gamma_k \tag{SM1.30}$$

$$\leq \sum_{k=1}^{q} \gamma_k \tag{SM1.31}$$

where (SM1.29) is due to $\ell \geq q_0$, (SM1.30) follows from (SM1.20)-(SM1.21) and (SM1.31) holds since $\gamma_q \geq \gamma_{q_0}$ for $q < q_0$.

($\Leftarrow$) The converse part is a direct consequence of Lemma SM1.2 and the direct part of the proof. Indeed, if test (SM1.2) is passed for some index $\ell \in [\![1, n]\!]$ then by virtue of Lemma SM1.2 there exists some $q_0 \in [\![1, n]\!]$ such that test (SM1.1) also passes. Now, from the proof of the direct part of the lemma, we have that test (SM1.2) is passed for index ℓ if and only if $\ell \in [\![q_0, n]\!]$.

REFERENCES

[SM1] J. Larsson, M. Bogdan, and J. Wallin, *The strong screening rule for SLOPE*, 2020, https://arxiv.org/abs/2005.03730.